VI. Concluding Remarks

In this correspondence, it has been shown that the Hankel matrix (1) defined by Newton's sums can be used to obtain information on the location of zeros of a real polynomial $f(x)$. In [6, pp. 208–236], certain other types of Hankel matrices constructed from Markov parameters of the real rational function $R=p(x)/q(x)$ have been used in connection with the computation of the Cauchy index of R and to prove Routh–Hurwitz stability criterion. Interesting relationships between these matrices and the Bezoutian matrix, defined by two polynomials $p(x)$ and $q(x)$, have been established by Anderson in [1] and [2]. It would be quite interesting to see if these Hankel matrices of Markov parameters are also related to the Hankel matrix (1) of Newton sums used in this correspondence.

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Weak and Strong Max-Min Controllability

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Abstract—**Weak and strong max-min controllability for a two player (time-varying) linear control system are defined. It is proved that the two concepts are equivalent.**

Consider the following linear system with dual controls:

$$\dot{x}=A(t)x+B_p(t)u+B_e(t)v, \qquad x(t_0)=x_0 \tag{1}$$

where $x=x(t)\in\mathbf{R}^n$ is the state vector with x_0 a specified initial state. The vectors $u=u(t)\in\mathbf{R}^{m_p}$ and $v=v(t)\in\mathbf{R}^{m_e}$, regarded, respectively, as the *pursuer* and *evader* controls are required to satisfy $\int_I \|u(t)\|^2\,dt<\infty$ and $\int_I \|v(t)\|^2\,dt<\infty$ on each compact interval $I\subset[t_0,\infty)$, where $\|\cdot\|$ denotes the Euclidean norm. The matrices A, B_p, and B_e are assumed to have entries which are real and measurable on $[t_0,\infty)$. For any pair of controls u and v we shall denote by $x(t)=\varphi(t,t_0,x_0,u,v)$ the corresponding unique solution of (1) starting at x_0 at time t_0 $(t\geqslant t_0)$.

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Definition 1: An event (t_0,x_0) in system (1) is *weakly max-min controllable* if for each announced evader control v on $[t_0,\infty)$. there exists a time $\bar{t}=\bar{t}(v)\geqslant t_0$ and a pursuer control u on $[t_0,\bar{t}]$ such that $x(\bar{t})=\varphi(\bar{t},t_0,x_0,u,v)=0$.

Definition 2: An event (t_0,x_0) in system (1) is *strongly max-min controllable at time T* $(T\geqslant t_0)$ if for each announced evader control v on $[t_0,T]$ there exists a pursuer control u on $[t_0,T]$ such that $x(T)=\varphi(T,t_0,x_0,u,v)=0$. The event (t_0,x_0) is *strongly max-min controllable* if it is strongly max-min controllable for some $T\in[t_0,\infty)$.

Obviously strong max-min controllability of an event implies weak max-min controllability. It will be shown in this note that the converse is also true. In [1] an extensive investigation of max-min controllability is presented.

It will be convenient to transform system (1) by a change of coordinates as follows: let $\Phi(t,t_0)$ denote the fundamental matrix solution corresponding to system (1) and define the vector function

$$z(t)=\Phi(t_0,t)x(t), \qquad t\geqslant t_0. \tag{2}$$

Then, it is readily verified that z satisfies the following differential equation:

$$\dot{z}=\tilde{B}_p(t)u-\tilde{B}_e(t)v, \qquad z(t_0)=x(t_0)=x_0 \tag{3}$$

where $\tilde{B}_p(t):=\Phi(t_0,t)B_p(t)$ and $\tilde{B}_e(t):=\Phi(t_0,t)B_e(t)$. Furthermore, it is easily noted that $z(t)=0$ if and only if $x(t)=0$, whence system (3) is completely equivalent in respect to max-min controllability to system (1). We shall, henceforth, be concerned with the latter system.

Define the controllability Grammians for the pursuer and evader by

$$W_p(t_0,t):=\int_{t_0}^{t}\tilde{B}_p(\sigma)\tilde{B}_p'(\sigma)\,d\sigma, \qquad t\geqslant t_0 \tag{4}$$

$$W_e(t_0,t):=\int_{t_0}^{t}\tilde{B}_e(\sigma)\tilde{B}_e'(\sigma)\,d\sigma, \qquad t\geqslant t_0 \tag{5}$$

where the prime denotes transpose. Clearly, W_p and W_e are symmetric nonnegative definite $(n\times n)$ matrices and it is easily verified that their rank is a nondecreasing and left-continuous function of time (the latter holding because $\mathfrak{R}(F)\subset\mathfrak{R}(F+G)$ for every pair of symmetric nonnegative matrices F,G where $\mathfrak{R}(\cdot)$ denotes range). In [1] the following theorem is proved.

Theorem 1: Given system (3) with $z_0\neq 0$, a necessary and sufficient condition for an event (t_0,z_0) to be strongly max-min controllable in finite time $T>t_0$ is that the following conditions hold:

$$z_0\in\mathfrak{R}(W_p(t_0,T)) \tag{6}$$

$$\mathfrak{R}(W_e(t_0,T))\subset\mathfrak{R}(W_p(t_0,T)). \tag{7}$$

The remainder of this note will be devoted to proving the following theorem.

Theorem 2: Consider system (3). An event (t_0,z_0) is strongly max-min controllable if and only if it is weakly max-min controllable.

We only have to prove that weak max-min controllability implies strong max-min controllability for some finite $T\geqslant t_0$.

Let

$$T_1:=\inf_{t\geqslant t_0}\{t: z_0\in\mathfrak{R}(W_p(t_0,t))\}.$$

Under the assumption of weak max-min controllability we obviously have that T_1 is finite, and due to Grammian monotonicity, $z_0\in\mathfrak{R}(W_p(t_0,t))$ for all $t>T_1$. Hence, if T_2 is defined by

$$T_2:=\sup_{t\geqslant t_0}\{t:\mathfrak{R}(W_e(t_0,t))\subset\mathfrak{R}(W_p(t_0,t))\},$$

it follows from Theorem 1 that (t_0,z_0) is strongly max-min controllable if and only if $T_2>T_1$. (It should be observed that when finite, T_1 and T_2

are points of discontinuity of rank $(W_p(t_0,t))$ and of rank $(W_e(t_0,t))$, respectively. Also, when T_2 is finite, Grammian left-continuity implies that the sup in the definition of T_2 is attained as a max.) The proof of Theorem 2 will be accomplished by showing that if $T_2 \leqslant T_1$ (i.e., when strong max-min controllability fails to hold), then there exists a control v^* against which no u can drive z_0 to the origin, thereby showing that weak max-min controllability also fails to hold.

We shall require the following lemmas.

Lemma 1: Given system (3) with $z_0 \neq 0$, assume that T_1 is finite and that $T_2 \leqslant T_1$. Let (u,v) be a pair of control functions such that the corresponding solution of (3) satisfies $z(\tilde{t})=0$ for some $t_0 < \tilde{t} < \infty$. Then $\tilde{t} > T_2$.

Proof: Suppose $\tilde{t} \leqslant T_2$. If $z(\tilde{t})=0$ then the control v must satisfy

$$z_0 - \int_{t_0}^{\tilde{t}} \tilde{B}_e(t)v(t)\,dt \in \mathcal{R}\big(W_p(t_0,\tilde{t})\big). \tag{8}$$

Since $\tilde{t} \leqslant T_2$ and since $\int_{t_0}^{\tilde{t}} \tilde{B}_e(t)v(t)\,dt \in \mathcal{R}(W_e(t_0,\tilde{t}))$, it follows that

$$\int_{t_0}^{\tilde{t}} \tilde{B}_e(t)v(t)\,dt \in \mathcal{R}(W_p(t_0,T_2)) \tag{9}$$

and

$$\mathcal{R}\big(W_p(t_0,\tilde{t})\big) \subset \mathcal{R}(W_p(t_0,T_2)). \tag{10}$$

From (8)–(10) we conclude that

$$z_0 \in \mathcal{R}(W_p(t_0,T_2)) \tag{11}$$

which in view of Grammian left-continuity violates the assumption that $T_2 \leqslant T_1$. □

Lemma 2: Given system (3), let $T_2 < \infty$ and let T_2' denote the first discontinuity of rank $W_e(t_0,t)$, $t > T_2$. Then for each positive $\epsilon < T_2' - T_2$ there exists a measurable set I_ϵ of positive measure, $I_\epsilon \subset [T_2, T_2+\epsilon)$, such that $\mathcal{R}(\tilde{B}_e(t)) \not\subset \mathcal{R}(W_p(t_0,T_2+\epsilon))$ for all $t \in I_\epsilon$.

Proof: Assume the lemma is false. Then for some $\epsilon_1 > 0$ we must have

$$\mathcal{R}(\tilde{B}_e(t)) \subset \mathcal{R}(W_p(t_0,T_2+\epsilon_1)) \tag{12}$$

for almost all $t \in [T_2, T_2+\epsilon_1)$. Since $\mathcal{R}(\tilde{B}_e(t)\tilde{B}_e'(t)) \subset \mathcal{R}(\tilde{B}_e(t))$, for all $t \in [T_2, T_2+\epsilon_1)$ and since $\mathcal{R}(\sum_{i=1}^k \alpha_i \tilde{B}_e(t_i)\tilde{B}_e'(t_i)) \subset \sum_{i=1}^k \mathcal{R}(\tilde{B}_e(t_i)\tilde{B}_e'(t_i))$ for all α_i real, $t_i \in [T_2, T_2+\epsilon_1)$, $i=1,2,\cdots,k$ where k is an arbitrary positive integer, we conclude from the definition of the Lebesgue integral that

$$\mathcal{R}\left(\int_{T_2}^{T_2+\epsilon_1} \tilde{B}_e(t)\tilde{B}_e'(t)\,dt\right) \subset \mathcal{R}(W_p(t_0,T_2+\epsilon_1)). \tag{13}$$

Clearly

$$\mathcal{R}(W_e(t_0,T_2+\epsilon_1)) \subset \mathcal{R}(W_e(t_0,T_2)) + \mathcal{R}(W_e(T_2,T_2+\epsilon_1)). \tag{14}$$

Since $\mathcal{R}(W_e(t_0,T_2)) \subset \mathcal{R}(W_p(t_0,T_2+\epsilon_1))$, and since by (13) also $\mathcal{R}(W_e(T_2,T_2+\epsilon_1)) \subset \mathcal{R}(W_p(t_0,T_2+\epsilon_1))$, (14) implies

$$\mathcal{R}(W_e(t_0,T_2+\epsilon_1)) \subset \mathcal{R}(W_p(t_0,T_2+\epsilon_1)) \tag{15}$$

thus contradicting the definition of T_2. □

Lemma 3: Given system (3), let $\tilde{T} \geqslant T_2$ be any point of discontinuity of rank $W_e(t_0,t)$, and let $\tilde{T}'$ be the first discontinuity of rank $W_e(t_0,t)$, $t \geqslant \tilde{T}$. Then for each positive $\epsilon < \tilde{T}' - \tilde{T}$ there exists a measurable set I_ϵ of positive measure $I_\epsilon \in [T_2, T_2+\epsilon]$, such that $\mathcal{R}(\tilde{B}_e(t)) \not\subset \mathcal{R}(W_p(t_0,T_2+\epsilon))$ for almost all $t \in [T_2, T_2+\epsilon)$.

The proof of Lemma 3 is similar to that of Lemma 2 and is therefore omitted.

Lemma 4: Let $z(t)$ be a measurable $\boldsymbol{R}^n$-valued function and let $\mathcal{W}$ be a proper subspace of $\boldsymbol{R}^n$. Let $I_\epsilon \subset \boldsymbol{R}^1$ be a measurable set of positive measure such that $z(t) \not\in \mathcal{W}$ for all $t \in I_\epsilon$. Then there exists a vector h such that $\langle h,w\rangle = 0$ for all $w \in \mathcal{W}$ and $\langle h,z(t)\rangle \neq 0$ for all $t \in I_\epsilon$.

Proof: Consider a hyperplane $\mathcal{H}$ such that $\mathcal{W} \subset \mathcal{H}$. Each vector $z(t)$ is either in one of the two open half spaces determined by $\mathcal{H}$ or in $\mathcal{H}$ itself. First assume that $z(t) \not\in \mathcal{H}$ almost everywhere. Then choose $h \in \mathcal{H}^\perp$. If the aforementioned assumption does not hold, one repeats the argument in an appropriate subspace of $\mathcal{H}$. □

Assume $T_2 \leqslant T_1$. We shall now turn our attention toward the construction of a control v^* such that against v^* no u can force z_0 to 0 in finite time. This of course will complete our proof of Theorem 2. First, we shall replace $\tilde{B}_p(t)$ with a measurable $(n \times p)$ matrix function $\bar{\tilde{B}}_p(t)$ satisfying the following properties.

Property 1: $\bar{\tilde{B}}_p(t) \equiv \tilde{B}_p(t)$, for all $t \in [t_0, T_2]$.

Property 2: $\mathcal{R}(W_p(t_0,t)) \subset \mathcal{R}(\overline{W}_p(t_0,t))$, for all $t \geqslant t_0$ where $\overline{W}_p$ denotes the Grammian associated with $\bar{\tilde{B}}_p$.

Property 3: $\overline{W}_p(t_0,t)$ is constant between consecutive discontinuities τ_1, τ_2 of rank $(W_e(t_0,t))$, $T_2 \leqslant \tau_1 < \tau_2$.

Property 4: At each discontinuity σ of rank $\overline{W}_p(t_0,t)$, $\sigma > T_2$, we have $\mathcal{R}(W_e(t_0,\sigma)) \subset \mathcal{R}(\overline{W}_p(t_0,\sigma+\epsilon))$, for all $\epsilon > 0$.

Property 5: $\mathcal{R}(\overline{W}_p(t)) \not\subset \mathcal{R}(W_e(t))$, for all $t > T_2$.

One can readily see that a $\bar{\tilde{B}}_p$ satisfying Properties 1–5 can always be built—one first specifies $\overline{W}_p$ and then defines an appropriate $\bar{\tilde{B}}_p$. Details of this elementary recursive (but somewhat tedious) construction are left to the reader. Consider now the system

$$\dot{z} = \bar{\tilde{B}}_p(t)u - \tilde{B}_e(t)v, \qquad z(t_0) = z_0. \tag{3'}$$

Observe that if we construct a control v^* such that no u can drive z_0 to 0 in system (3′) then (by Property 2) this is certainly the case for system (3), which is what we require. Upon defining $\overline{T}_1$ for system (3′) analogously to T_1 for system (3), it follows that $T_2 \leqslant \overline{T}_1 \leqslant T_1$. The required control v^* is now constructed by the following procedure.

1) On (t_0, T_2) apply an arbitrary evader control. Lemma 1 guarantees the impossibility of $z(t)=0$ at or before T_2.

2) Let T_2' denote the first discontinuity of rank $W_e(t_0,t)$, $T_2' > T_2$. If $T_2 < \overline{T}_1$ then take $v^*=0$ on $[T_2, T_2')$. Letting $\epsilon = (T_2' - T_2)/2$, it follows that in the presence of any u the solution of (3′) satisfies $z(t) \in z_0 + \mathcal{R}(\overline{W}_p(t_0,T_2+\epsilon))$ for all $t \in [T_2, T_2')$. $T_2 < \overline{T}_1$ implies then that $z_0 \not\in \mathcal{R}(\overline{W}_p(t_0,T_2+\epsilon))$, and thus $z(t) \neq 0$ for all $t \in [T_2, T_2')$. Now let $T_2 = \overline{T}_1$. Suppose $\mathcal{R}(\overline{W}_p(t_0,t)) \neq R^n$ for all t, for otherwise (6) and (7) hold for some T and there is nothing to prove. Due to Lemmas 2–4 it is readily seen that there exists a set I_ϵ of positive measure, $I_\epsilon \in [T_2, T_2+\epsilon)$, a measurable $\boldsymbol{R}^{m_e}$-valued function v on $[T_2, T_2+\epsilon)$, and a vector $h \in \mathcal{N}(\overline{W}_p(t_0,T_2+\epsilon))$ [$\mathcal{N}(\cdot)$ denoting null space] such that $\langle h, \tilde{B}_e(t)v(t)\rangle \neq 0$ for $t \in I_\epsilon$. Now let $\langle h, \tilde{B}_e(t)v^*(t)\rangle := \max_{\|v\| \leqslant 1} \langle h, \tilde{B}_e(t)v\rangle$. The maximizing control $v^*(t)$ on $[T_2, T_2+\epsilon)$ is measurable (see e.g., [2, p. 160]) and $\langle h, \tilde{B}_e(t)v^*(t)\rangle > 0$, for all $t \in I_\epsilon$. Hence, $\int_{T_2}^{T_2+t} \langle h, \tilde{B}_e(\tau)v^*(\tau)\rangle d\tau > 0$, for all $0 < t < \epsilon$, and so $\int_{T_2}^{T_2+t} \tilde{B}_e(\tau)v^*(\tau)\,d\tau \not\in \mathcal{R}(\overline{W}_p(t_0,T_2+\epsilon))$, for all $t \in (0,\epsilon)$. Now $z(t) \in z_0 + \int_{T_2}^{t} \tilde{B}_e(\tau)v^*(\tau)\,d\tau + \mathcal{R}(\overline{W}_p(t_0,T_2+\epsilon))$, for all $t \in [T_2, T_2')$. Hence, $z(t) \neq 0$, for all $t \in [T_2, T_2')$ in the presence of any evader control.

3) At discontinuities of rank $(W_e(t_0,t))$ beyond T_2', reapply 2) with Lemma 3 replacing the role of Lemma 2.

Since when played against v^* constructed above in 1)–3) no pursuer control can drive z_0 to 0, weak max-min controllability is violated. Hence, $T_2 > T_1$, concluding the proof of Theorem 2.

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